On a variational approach to truncated problems of moments.

C.-G. Ambrozie*, Prague

Abstract

We characterize the existence of the L^1 solutions of the truncated moments problem in several real variables on unbounded supports by the existence of the maximum of certain concave Lagrangian functions. A natural regularity assumption on the support is required.

Keywords: problem of moments, representing measure

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1 Introduction

The present paper is concerned with the truncated problem of moments in several real variables, in the following context. Let $n \in \mathbb{N}$ and fix a closed subset $T \neq \emptyset$ of $\mathbb{R}^n$, a finite subset $I \subset (\mathbb{Z}_+)^n$ with $0 \in I$ and a set $g = (g_i)_{i \in I}$ of real numbers with $g_0 = 1$, where $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Typically a problem of moments [1] requires to establish if there exist Borel measures $\nu \geq 0$ on $\mathbb{R}^n$, supported on T, such that $\int_T |t^i| d\nu(t) < \infty$ and $\int_T t^i d\nu(t) = g_i$ for all $i \in I$. As usual $t^i = t_1^{i_1} \cdots t_n^{i_n}$ where $t = (t_1, \ldots, t_n)$ is the variable in $\mathbb{R}^n$ and $i = (i_1, \ldots, i_n)$ is a multiindex. In this case we call ν a representing measure of g, and g_i the moments of ν. We are interested in those measures $\nu = f dt$ that are absolutely continuous with respect to the n-dimensional Lebesgue measure $dt = dt_1 \cdots dt_n$, in which case we call f a representing density of g. Namely the (class of equivalence of the) Lebesgue integrable function f is ≥ 0 almost everywhere (a.e.) on T, has finite moments of orders $i \in I$ and

$$\int_T t^i f(t) dt = g_i \quad (i \in I). \tag{1}$$

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Given partial information in the integral form $\int_T t^i \, \mathrm{f} \, \rho dt = g_i$ about representing densities f on a probability space $(T, \rho dt)$, endowed with a reference density ρ, does not determine them uniquely. An approach favorite to physicists and statisticians is to choose that particular density f_*, minimizing the entropy functional $h(\mathrm{f}) = \int_T (\mathrm{f} \ln \mathrm{f}) \rho \, dt$ amongst all solutions of the moments constraints. This uniquely selects the unbiased probability distribution f_* (that proves to have the form $\mathrm{f}_*(t) = e^{\sum_{i \in I} \lambda_i^* t^i}$) on the knowledge of the prescribed average values g_i of t^i, where t is considered as a T-valued random variable with repartition ρ [6], [9], [18], [20]. Under suitable hypotheses, f_* turns to exist, even for measures more general than ρdt. A main tool to this aim is Fenchel duality [8], [24], [26], [27], that deals with minimizing convex functions $h : X \to \mathbb{R} \cup \{\infty\}$ on convex subsets of locally convex spaces X, in connection with the dual problem of maximizing $-h^*$, where $h^* : X^* \to \mathbb{R} \cup \{\infty\}$ is the convex conjugate of h, called also its Legendre-Fenchel transform [26], [27], defined on the dual X^* of X by $h^*(y) = \sup\{\langle x, y \rangle - h(x) : h(x) < \infty\}$. Typically $\inf h = \max(-h^*)$ and, briefly speaking, minimizing $\int_T \mathrm{f} \ln \mathrm{f} \rho dt$ as above is to find $\lambda^* = (\lambda_i^*)_{i \in I}$ maximizing $L(\lambda) = \sum_{i \in I} g_i \lambda_i - \int_T e^{\sum_{i \in I} \lambda_i t^i} \rho dt$. Many results exist in this direction [3], [5] – [9], [16], [17], [21] – [23]. Additional hypotheses are always necessary when the conclusion $\inf h = \min h$ is sought for, since there are g for which the primal attainment fails [16], [17] although problem (1) has solutions.

By Theorem 3 we prove that the feasibility of problem (1) is equivalent to the boundedness from above $\sup L < \infty$ with attainment $\sup L = \max L$ for the concave Lagrangian function L. This holds no matter whether $\inf h$ is attained or not (the general theory still provides us with $\inf h = \max L$).

Initiated by Stieltjes, Hausdorff, Hamburger and Riesz, the area of the truncated problems of moments nowadays knows various other approaches, based for instance on operator methods or sums-of-squares representations for positive polynomials [10] – [14], [19], [25]. Although important, these topics remain beyond the aim of this work, focused on our mentioned Theorem 3.

The author got the idea to consider L instead of h from the works [5] where a similar characterization exists, and [16], [17], drawn to his attention by professor Mihai Putinar. Our statement and proof are rather general, independent of these cited works.

2 Main results

We remind that a linear Riesz functional φ_γ [12] associated to a set $\gamma = (\gamma_i)_{i\in J}$ of real numbers γ_i for $J \subset \mathbb{Z}_+^n$ is defined on the polynomials p from the linear span of $X_1^{i_1}\dots X_n^{i_n}$ where $i=(i_1,\dots,i_n)\in J$ by $\varphi_\gamma X^i = \gamma_i$. One calls φ_γ T-positive [12] if $\varphi_\gamma p \geq 0$ whenever $p(t)\geq 0$ for all $t\in T$. If γ has representing measures $\nu \geq 0$ on T, φ_γ is T-positive since $\varphi_\gamma p = \int_T p\,d\nu$ for any such polynomial p. In the full case $J=\mathbb{Z}_+^n$ the T-positivity condition is sufficient for the existence of the representing measures, by the Riesz-Haviland theorem [15]. An analogue of this theorem [12] for the truncated case $I=\{i : |i|\leq 2k\}$ characterizes the existence of the representing measures by the existence of T-positive extensions of φ_γ to the space of polynomials of degree $\leq 2k+2$. For later use, we state below a version of these results (Theorem 1) and a Fenchel theoretic result of dual attainment (Theorem 2).

Definitions We call T *regular* [4] if for any $t\in T$ and $\varepsilon>0$ the Lebesgue measure of the set $\{x\in T : \|x-t\|<\varepsilon\}$ is positive. As usual $\|t\| = (\sum_{\iota=1}^n t_\iota^2)^{1/2}$. For any $i\in I$ set $\sigma_i=\{j\in\mathbb{Z}_+^n : j_k = \text{either } 0 \text{ or } i_k,\ 1\leq k\leq n\}$. We call I *regular* [4] if $\sigma_i\subset I$ for all $i\in I$. Define Γ, $G\subset\mathbb{R}^N$ ($N=\operatorname{card} I$) by $\Gamma=\{\gamma=(\gamma_i)_{i\in I} : \exists \text{ measures } \nu\geq 0 \text{ on } T \text{ with } \int_T t^i d\nu(t)=\gamma_i,\ i\in I\}$ and $G=\{\gamma=(\gamma_i)_{i\in I}\neq 0 : \exists\ f\in L_+^1(T,dt) \text{ such that } \int_T t^i f(t)dt=\gamma_i,\ i\in I\}$. The notation $L^p(T,\mu)$, $L^p(\mu)$ for μ measure on T, $1\leq p\leq\infty$ has the usual meaning. In particular $L_+^1(T,\mu)$ is the set of all $f\in L^1(T,\mu)$, $f\geq 0$ μ-a.e. For $\gamma=(\gamma_i)_{i\in I}$, φ_γ is the linear functional defined on the span $P_I\subset\mathbb{R}[X_1,\dots,X_n]$ of all X^i with $i\in I$ by $\varphi_\gamma X^i=\gamma_i$. Set $\mathrm{e}_\iota=(0,\dots,\overset{\iota}{1},\dots,0)$ for $1\leq\iota\leq n$.

By [Theorem 6,[4]] the convex cone G is the dense interior of the cone Γ.

Theorem 1 [Theorem 7,[4]] *Let $T\subset\mathbb{R}^n$ be a closed regular set, $I\subset\mathbb{Z}_+^n$ a finite regular set and $g=(g_i)_{i\in I}$ a set of numbers with $g_0=1$. Then $g\in G$ $\Leftrightarrow$ $\varphi_g\, p>0$ for every $p\in P_I\setminus\{0\}$ such that $p(t)\geq 0$ for all $t\in T$.*

Theorem 2 [Corollary 2.6,[8]] *Let $\mathcal{T}$ be a space with finite measure $\mu\geq 0$, $1\leq p\leq\infty$ and $a_i\in L^q(\mu)$, $g_i\in\mathbb{R}$ for $i\in I=$ finite where $\frac{1}{p}+\frac{1}{q}=1$. Let $\phi:\mathbb{R}\to(-\infty,\infty]$ be proper, convex, lower semicontinuous with $\phi|_{(0,\infty)}<\infty$. If there are $x\in L^p(\mu)$, $x>0$ a.e. such that $\phi\circ x\in L^1(\mu)$ and $\int_{\mathcal{T}} a_i\,x\,d\mu=g_i$, then the quantities*

$$P=\inf\{\int_{\mathcal{T}}\phi(x(t))d\mu(t) : x\in L^p(\mu), x\geq 0 \ a.e.,\ \phi\circ x\in L^1(\mu), \int_{\mathcal{T}} a_i x d\mu = g_i\ \forall i\},$$

$$D=\max\{\sum_{i\in I} g_i\lambda_i - \int_T \phi^*(\sum_{i\in I}\lambda_i a_i(t))\,d\mu(t) : \lambda_i\in\mathbb{R},\ \phi^*\circ\sum_{i\in I}\lambda_i a_i \in L^1(\mu)\,\}$$

are equal, $-\infty \le P = D < \infty$ *and the maximum* D *is attained.*

Theorem 3 is a reminiscent to [Theorem 4, [3]], where $\int_T \mathrm{f}\ln\mathrm{f}\rho dt$ is minimized subject to $\int_T t^i \mathrm{f}\rho dt = g_i$ under stronger hypotheses on ρ, like $\rho(t) \sim e^{-\varepsilon\|t\|^p}$ with $p > 2k$ (to fit the notation in [3], let $a = 1$ and our $f := \rho\,\mathrm{f}$, whence $L_{\rho,a,g}(\lambda) = L(\lambda-\lambda_0)+1$, with $\lambda_0 = (\lambda_{0i})_{i\in I}$ where $\lambda_{0i} = \delta_{i,0}$ and $\delta_{i,j}$ is Kronecker's symbol, $\delta_{i,j} = 1$ if $i = j$ and 0 if $i \neq j$). Although we do not obtain here the existence of a maximum entropy solution f_*, our present hypothesys on ρ are weaker, while condition $g \in G$ still characterized in Lagrangian terms. Our proof below relies on Theorem 1 ([Theorem 7,[4]]) and Theorem 2 ([Corollary 2.6,[8]]).

Theorem 3 *Let* $T \subset \mathbb{R}^n$ *be a closed regular set. Let* $I \subset \mathbb{Z}^n_+$ *be a finite regular set such that* $\max_{i\in I}|i| = 2k$ *where* $k\in\mathbb{N}$. *Assume* $2k\mathrm{e}_\iota \in I$ $(1\le\iota\le n)$. *Let* $g = (g_i)_{i\in I}$ *be a set of numbers with* $g_0 = 1$. *Fix* $\rho \in L^1(T, dt)$, $\rho > 0$ *a.e. The following statements* (a) *and* (b) *are equivalent:*

(a) *There exist functions* $f \in L^1_+(T, dt)$ *such that* $\int_T |t^i| f(t)dt < \infty$ *and*

$$\int_T t^i f(t)dt = g_i \quad (i \in I);$$

(b) *The functional* $L : \mathbb{R}^N \to \mathbb{R}\cup\{-\infty\}$ *defined by*

$$L(\lambda) = \sum_{i\in I} g_i\lambda_i - \int_T e^{\sum_{i\in I}\lambda_i t^i}\rho(t)dt, \qquad \lambda = (\lambda_i)_{i\in I}$$

is bounded from above and $\sup L$ *is attained in a (unique) point* λ^*.

Proof. Since $L(0) > -\infty$, $L \not\equiv -\infty$. Since $g_0 = 1$, each of the conditions (a) and (b) implies that T has positive Lebesgue measure, finite or not. Hence by means of Jensen's inequality one can show that L is strictly concave. Then whenever $\sup L$ is finite and attained at some point λ^*, this λ^* is unique.

(a) $\Rightarrow$ (b) The regularity condition on T is not necessary for this implication. Let $\mu = \tilde\rho dt$ be the measure on T with density $\tilde\rho := \rho e^{-\sum_{\iota=1}^n t_\iota^{2k}}$. Then $0 < \mu(T) < \infty$. Since (1) has a solution f, then $\tilde f := f/\tilde\rho$ satisfies

$$\int_T t^i \tilde f(t)\,d\mu(t) = g_i \quad (i \in I). \tag{2}$$

By [Theorem 2.9, [8]], see also [Lemma 4, [4]] for $\beta = 0$, problem (2) has also a solution $f_0 \in L^\infty(T)$ with $f_0 > 0$ a.e. The conclusion $\sup L < \infty$ may hold either directly by Theorem 2, or by an elementary argument as shown below. Let $x = f_0(t)$ a.e. and $y = \|f_0\|_\infty + 1$ in the inequalities $-e^{-1} \le x \ln x \le y \ln y$ for $0 \le x \le y$, $y \ge 1$, then integrate with respect to μ. Hence $f_0 \ln f_0 \in L^1(T, \mu)$. Fix $\lambda = (\lambda_i)_{i \in I}$. Let $x = f_0(t)$ and $y = \sum_{i \in I} \lambda_i t^i$ in the simple version $x \ln x - x \ge xy - e^y$ of Fenchel's inequality [27], then integrate. It follows, using (2) for f_0, that

$$\int_T f_0 \ln f_0 d\mu - \int_T f_0 d\mu \ge \sum_{i \in I} g_i \lambda_i - \int_T e^{\sum_{i \in I} \lambda_i t^i} d\mu(t) = L(\lambda - \lambda_0) + \sum_{i \in I} g_i \lambda_{0i}$$

where $\lambda_0 = (\lambda_{0i})_{i \in I}$ with $\lambda_{0i} = \sum_{\iota=1}^n \delta_{i,\, 2k e_\iota}$ and $\delta_{i,j}$ is Kronecker's symbol. Since λ was arbitrary, we get $\sup_\lambda L(\lambda) < \infty$. Now for the attainment $\sup L = \max L$, we need Theorem 2 as follows. Use $|t_j| \le (\sum_{\iota=1}^n t_\iota^{2k})^{1/2k}$,

$$|t^i| = |t_1|^{i_1} \cdots |t_n|^{i_n} \le (\sum_{\iota=1}^n t_\iota^{2k} + 1)^{|i|/2k} \le \sum_{\iota=1}^n t_\iota^{2k} + 1 \qquad (|i| \le 2k)$$

and $\nu + 1 \le e^\nu$ for $\nu = \sum_{\iota=1}^n t_\iota^{2k}$ to get $\int_T |t^i| d\mu(t) \le \int_T \rho dt < \infty$ for $i \in I$. Then let: $\mathcal{T} = T$, the measure $\mu = \tilde{\rho} dt$, $p = \infty$, the moment functions $a_i(t) = t^i$ and the integrand ϕ be defined by $\phi(x) = x \ln x$ for $x > 0$, $\phi(0) = 0$ and $\phi(x) = +\infty$ for $x < 0$. The feasibility hypotheses is fulfilled by $x = f_0$. The convex conjugate $\phi^*(y) = \sup_{x \ge 0}(xy - x \ln x)$ of ϕ is given by $\phi^*(y) = e^{y-1}$ for $y \in \mathbb{R}$. We get the attainment $D = \sup \mathcal{L}$ for $\mathcal{L}(\lambda) = L(\lambda - \lambda_0') + \sum_{i \in I} g_i \lambda_{0i}'$ where $\lambda_0' = (\lambda_{0i}')_{i \in I}$ with $\lambda_{0i}' = \lambda_{0i} + \delta_{i,0}$. Thus we obtain a λ^* such that $\sup L = L(\lambda^*)$.

(b) $\Rightarrow$ (a) Let $\lambda^* \in \mathbb{R}^N$ such that $\sup L = L(\lambda^*)$. We prove that φ_g satisfies the positivity condition in Theorem 1. Let $p = \sum_{i \in I} \lambda_i X^i$, $p \not\equiv 0$ be arbitrary such that $p(t) \le 0$ for $t \in T$. The vector $\lambda := (\lambda_i)_{i \in I}$ is then $\ne 0$. For any $r > 0$, set $e_r(t) = e^{r \sum_{i \in I} \lambda_i t^i}$. Thus $e_r(t) \le 1$ for $t \in T$. Then the integral term $\int_T e_r \rho dt$ of $L(r\lambda) = r \sum_{i \in I} g_i \lambda_i - \int_T e_r \rho\, dt$ remains bounded as $r \to \infty$. Hence $\varphi_g p = \sum_{i \in I} g_i \lambda_i \le 0$, for otherwise the linear term $r \varphi_g p$ of $L(r\lambda)$ would give $\sup L = \infty$ that is false. Assume that $\varphi_g p = 0$. Then the restriction of the function L to the half-line $\ell := \{r\lambda : r > 0\}$ is given by the function $r \mapsto -\int_T e_r \rho dt$. This function is finite, bounded and strictly monotonically increasing on $(0, \infty)$. Use to this aim that $0 < e_r \le 1$, $\int_T \rho dt < \infty$, $e_r = e^{rp}$ with $p \le 0$ and $L|_\ell$ is strictly concave. Then a finite

limit $\lim_{r\to\infty} L(r\lambda) = \sup_{\ell} L$ exists, in particular $\sup_{r\geq 1} |L(r\lambda)| < \infty$. For $a>0$,

$$\infty > L(\lambda^* + a\lambda) = \sum_{i\in I} g_i\lambda_i^* + a\sum_{i\in I} g_i\lambda_i - \int_T e^{\sum_{i\in I}\lambda_i^* t^i} e^{a\sum_{i\in I}\lambda_i t^i}\rho(t)dt$$

$$\geq \sum_{i\in I} g_i\lambda_i^* + r\cdot 0 - \int_T e^{\sum_{i\in I}\lambda_i^* t^i}\rho(t)dt = L(\lambda^*) = \max L \geq L(0) > -\infty$$

because $\sum_{i\in I} g_i\lambda_i = 0$ and $\sum_{i\in I}\lambda_i t^i \leq 0$ for all $t \in T$. Hence L is finite on every point of the half-line $\{\lambda^* + a\lambda\}_{a>0}$. Note that λ^* cannot be colinear with λ due the behaviour of L on ℓ: firstly, $\lambda^* \notin \ell$ because L reaches its global maximum only in λ^* while $L|_\ell$ increases strictly along ℓ as $r \to \infty$. Also $\lambda^* \notin \{0\} \cup (-\ell)$, for otherwise the concavity of the restriction $L|_{\mathbb{R}\lambda} : \mathbb{R}\lambda \to \{-\infty\} \cup \mathbb{R}$ of L to the line $\mathbb{R}\lambda$ would imply, for some $r \geq 0$ with $\lambda^* = -r\lambda$, that $L(r\lambda) \geq L(0) = L(\frac{1}{2}(\lambda^* + r\lambda)) \geq \frac{1}{2}(L(\lambda^*) + L(r\lambda))$, whence $L(\lambda^*) \leq L(r\lambda) < \sup L|_\ell \leq \sup L = L(\lambda^*)$ that is impossible. Thus $\lambda^* \notin \mathbb{R}\lambda$. Then a 2-dimensional drawing shows that for every $r > 1$ there is a unique point x_r of intersection of the segments $(\lambda^*, r\lambda)$ and $(\lambda, \lambda^* + \lambda)$. Write to this aim $x_r = s\lambda^* + (1-s)r\lambda = s'\lambda + (1-s')(\lambda^* + \lambda)$ with coefficients $s = s_r$, $s' = s'_r$, use the linear independence of λ^*, λ and get $s = (r-1)/r$, $s' = 1-s$ whence s, $s' \in (0,1)$ and $\lim_{r\to\infty} s'_r = 0$. Then $\lim_{r\to\infty} x_r = \lambda^* + \lambda$. The concavity (and hence, continuity [27]) of L on the segment $(\lambda, \lambda^* + \lambda]$ gives $\lim_{r\to\infty} L(x_r) = L(\lambda^* + \lambda) < L(\lambda^*)$ with strict inequality, because the point λ^* of maximum of L is unique. But $L(x_r) = L(s\lambda^* + (1-s)r\lambda) \geq sL(\lambda^*) + (1-s)L(r\lambda)$ and letting $r \to \infty$ we derive, using $\lim_{r\to\infty} s_r = 1$ and $\sup_{r\geq 1}|L(r\lambda)| < \infty$, that $\lim_{r\to\infty} L(x_r) \geq L(\lambda^*)$. We got a contradiction. Then $\varphi_g p < 0$. The feasibility of problem (1) follows then by Theorem 1. $\square$

Remarks Since λ^* may be on the boundary of $\operatorname{dom} L := \{\lambda : L(\lambda) > -\infty\}$, one cannot prove (b) $\Rightarrow$ (a) by derivating under the integral in λ^*, and the h-minimization may fail [17]. Additional hypotheses may compel λ^* to be interior to $\operatorname{dom} L$ [16] in which case the entropy minimization can be obtained [24], providing the particular solution $f_*(t) = e^{\sum_{i\in I}\lambda_i^* t^i}$, see for instance [3]. For example let $T = \mathbb{R}^n$, $I = \{i : |i| \leq 2k\}$ and $\rho(t) = e^{-\|t\|^{2k}}$. By Theorem 3, problem (1) is feasible if and only if L is bounded from above and attains its maximum in a point λ^*, even when a minimum entropy solution does not exist. By Fatou's lemma and Lebesgue's dominated convergence theorem, $f_0 := e^{\sum_{|i|\leq 2k}\lambda_i^* t^i}$ has finite moments of order $\leq 2k$, we can get $\int t^i f_0 dt = g_i$

for $|i| < 2k$ and $\int t_\iota^{2k} f_0 dt \leq g_{2ke_\iota}$ $(1 \leq \iota \leq n)$, but the equalities (1) may fail for $|i| = 2k$ [17]. By integration in polar coordinates, the homogeneous polynomial $p := \sum_{|i|=2k} \lambda_i^* X^i$ is shown to always satisfy $p(t) \leq 0$ on $\mathbb{R}^n$; if moreover $p(t) < 0$ for all $t \neq 0$, then λ^* is interior to $\operatorname{dom} L$ and f_0 is indeed a solution of problem (1), $f_0 = f_*$. We omit the details and refer the reader to [16], [17].

Note also that whenever ρ is at our disposal, various choices may be tried [3] to facilitate the numerical maximization of $L = L_\rho$.

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Institute of Mathematics, AS CR
Zitna 25
115 67 Prague 1
Czech Republic
ambrozie@math.cas.cz

and: Institute of Mathematics "Simion Stoilow" - Romanian Academy, PO Box 1-764, 014700 Bucharest, Romania